Technique to find Pareto-optimal solutions to multiple objective linear programming problems with intuitionistic fuzzy goals and its application on production industry

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Abstract A technique to find Pareto-optimal solutions to multiple objective linear programming problems under intuitionistic fuzzy environment is presented in this paper. In 1997, Angelov proposed optimization technique under intuitionistic fuzzy environment. In 2009, Jimenez and Bilbao showed that a fuzzy efficient solution may not be Pareto-optimal solution in case that one of the fuzzy goals is fully achieved. In 2015, Wu, Liu and Lur redefined membership function of fuzzy set theory and proposed another two phase technique to find Pareto-optimal solution. In this paper it is observed that strictly monotonic part of both membership and non-membership functions of intuitionistic fuzzy goals are useful in optimization technique. Further, better optimal solutions may be found if constraints that sum of minimal level of acceptance and maximal level of rejection not exceeding unity as well as minimal level of acceptance not exceeding unity are removed. Moreover few such constraints, used in existing techniques, may make a problem infeasible. Consequently, new functions: $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions are defined in place of membership function and non-membership function respectively in intuitionistic fuzzy decision making and used in proposed algorithm. The concept of Pareto-optimality in intuitionistic fuzzy environment and new method to test Pareto-optimality of solution is proposed in this paper. Necessary counter examples are given and one application on production industry further illustrates proposed algorithm. Conclusions are drawn at last.

Keywords Intuitionistic fuzzy optimization · I-Pareto-optimal solution · Multi-objective optimization · Pareto-optimal solution · Production planning · $T^{(+)}$-characteristic function · $T^{(-)}$-characteristic function.

1 Introduction

In literature, the framework of multiple objective linear programming problems (MOLPP) with k objective functions $z_i(x) = c_i x, i = 1 \dots k$ is usually considered as the following model

$$\begin{aligned} &Min\ \mathrm{z(x)} = (\mathrm{z}_1(x), \mathrm{z}_2(x), \dots, \mathrm{z}_k(x))^T = (c_1 x, c_2 x, \dots, c_k x)^T \\ &subject\ \mathrm{to}\ \mathrm{x} \in \mathrm{X} \end{aligned} \tag{1}$$

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where $X = \{x \in R^n : Ax \le b, x \ge 0\}, b = (b_1, b_2, ..., b_m) \in R^m$ and A is a $m \times n$ matrix and R denotes the set of real numbers. By assuming that the decision maker (DM) has imprecise aspiration levels for each of the objective functions $z_i(x) = c_i x$, i = 1. . .k of model (1), several methods have been proposed in literature to characterize Pareto-optimal solutions to MOLPP (1).

In one such approach, fuzzy set theory is used [see, Bellman and Zadeh 1970]. Fuzzy optimization technique is more flexible than deterministic optimization technique and allows us to find solutions that are more adequate to real life. And one popular method for finding Pareto-optimal solution under fuzzy environment is two phase method. Here max-min operator is usually applied due to its easy computation but it is not guaranteed to yield Pareto-optimal solution if more than one optimal solution exists. Using the linear membership functions of fuzzy objective functions (and constraints till they are symmetric), max-min operator proposed by Zimmermann (1978) can be applied to solve model (1) and find the maximum membership grade that each of the membership functions can simultaneously attain. Thus model (1) can be converted into crisp linear programming problem as follows [see, e.g. Tanaka et al 1974, H. Zimmermann 1985]

$$\begin{aligned} &\max \quad \lambda \\ &\text{subject to} \\ &\mu_i(z_i(x)) \ge \lambda, \ \text{i} = 1\ldots\text{k}, \\ &0 \le \lambda \le 1, \ \text{x} \in \text{X}. \end{aligned} \tag{2}$$

There exist several methods in literature to obtain optimal solution to this problem (2) [see, Guu and Wu 1999]. Sakawa (1987) contained a rather comprehensive survey regarding to interactive methods for MOLPP and fuzzy MOLPP [see Sakawa 2013]. Dubois and Fortemps (1999) proposed a multi-step procedure instead of two phase method to solve model (2). Recently Jimenez and Bilbao (2009) suggested that fuzzy-efficient solution may not be Pareto-optimal solution in case that one of fuzzy goals is fully achieved. And their proposed procedure extended two-phase approach of Guu, Wu (1997, 1999) and approach of Dubois, Fortemps (1999) to attain Pareto-optimal solution to MOLPP under fuzzy environment. But according to Yan-Kuen Wu et al (2015), proposed approach by Jimenez and Bilbao (2009) cannot guarantee to be one general procedure to attain Pareto-optimal solution to model (1).

Yan-Kuen Wu et al (2015) redefined membership function of fuzzy set theory. But that definition of redefined membership function $\mu_i(c_i x) \forall i = 1\ldots k$ and its usage in mathematical model as well as given numerical examples in their paper are not analogous. It is clear that only part of redefined membership function $\mu_i(c_i x) \forall i = 1\ldots k$ where value of objective function does not exceed sum of goal and tolerance (for minimization type of objective functions) is used in mathematical model and in given numerical examples. But these were not added to set of existing constraints. Thus in problem formulation, redefined membership function $\mu_i(c_i x) \forall i = 1\ldots k$ is strictly monotone where as it is not so in definition. Only upper bound of membership function was removed but there still exists lower bound of membership function at zero [see, Wu et al 2015]. As already observed by us, in many cases, if the lower bound of membership function remains intact, the problem may become infeasible.

Deep analysis by us further highlights the cause to lie in definition of membership function of fuzzy set itself. Membership function of fuzzy set provides satisficing result when extreme ends of imprecise information can be quantified within boundary of zero and one. But it may not be always logically correct to quantify or measure each kind of imprecise information within one bounded subset of real line.

Consider fuzzy set $\tilde{A}$ of all countries that may win next Soccer World Cup in 2018. Most of us agree with the prediction that Japan will not be the champion. So it can be safely stated that Japan is in $\tilde{A}$ with membership value 0. Again chance of India's winning is also nil. So India is also in $\tilde{A}$ with membership value 0. It means that Japan and India are at same level! But it is certain that even if Japan wins 2018 Soccer World Cup, unfortunately India will not be able to participate and hence she will certainly not be the winner. So membership function of fuzzy set theory fails to explain this case!

An element in fuzzy set *may lie partly or never lie or must lie* in that set. The idea of lying, lying partly or not lying is well quantified / measured by membership function of fuzzy set theory. But it does not suit in case of must lying (membership value in fuzzy set is always one), and never lying (membership value in fuzzy set is always zero).

On the other hand, fuzzy set theory has been widely developed and several new modifications and generalizations have appeared. One of them is the concept of intuitionistic fuzzy (IF) sets, introduced by K. T. Atanassov (1986). It considers not only degree of membership $\mu_i(z_i(x)),\ \mathrm{i}=1\ldots\mathrm{k}$ but also degree of non-membership $\nu_i(z_i(x)),\ \mathrm{i}=1\ldots\mathrm{k}$, such that sum of these values does not exceed unity to each IF objective. So MOLPP (1) may be treated as MOLPP under IF environment as well. And analogous to above discussions, in IF sets, introduction of two new functions in place of both membership functions and non-membership functions are necessary.

The rest of the paper is organized as follows: Sect. 2 introduces definitions. Sect. 3 highlights loopholes of existing technique and suggests remedy to find Pareto optimal solutions to MOLPP under IF environment. Sect. 4 proposes related algorithm. Sect. 5 presents numerical examples where existing technique fails and proposed methods gives acceptable solution. Sect. 5 also presents an application on purchasing bulk raw materials of a large-scale integrated steel plant. Sect. 6 gives concluding remarks.

2 Definitions

It is observed that an element of an IF set can lie partly or never lie or must lie in that set. Moreover in case of decision making under IF environment, one interesting and useful property of membership function is that higher value of membership function always gives better result of objective function; and analogous property of non-membership function is that lower value of non-membership function gives better result of objective function. *And usual membership functions, non-membership functions are not the only candidates having those amazing characteristics.* Hence in IF environment, new functions viz. $\mathrm{T}^{(+)}$-characteristic function in place of membership function and $\mathrm{T}^{(-)}$-characteristic function in place of non-membership function are defined.

Definition 1

Let S denotes the universal set and $\tilde{A}^i$ be an IF subset of S. Then $\mathrm{T}^{(+)}$-characteristic function is denoted by $T_{\tilde{A}^i}^{(+)}(x)$ and is defined as $T_{\tilde{A}^i}^{(+)}: S \to R$ that assigns a real number $T_{\tilde{A}^i}^{(+)}(x)$ to each element $x \in S$, here $T_{\tilde{A}^i}^{(+)}(x)$ represents degree of membership or acceptance level of $x \in S$ in $\tilde{A}^i$. And $\mathrm{T}^{(-)}$-characteristic function is denoted by $T_{\tilde{A}^i}^{(-)}(x)$ and is defined as $T_{\tilde{A}^i}^{(-)}: S \to R$ that assigns a real number $T_{\tilde{A}^i}^{(-)}(x)$ to each element $x \in S$, here $T_{\tilde{A}^i}^{(-)}(x)$ represents degree of non-membership or rejection level of $x \in S$ in $\tilde{A}^i$.

In model (1) it is usually not possible that all objective functions will simultaneously attain optimal values, due to their conflicting nature under IF environment. Concept of optimal solution is thus replaced by I-Pareto-optimal solution to MOLPP (1) under IF environment and consequently Pareto-optimal solution.

Definition 2

A decision plan $x_0 \in X$ is said to be an I-Pareto-optimal solution to the MOLPP (1) under IF environment if there does not exist another $y \in X$ such that $T_i^{+}(z_i(x_0)) \le T_i^{+}(z_i(y)),\ T_i^{-}(z_i(y)) \le T_i^{-}(z_i(x_0)),\ \forall i, i \ne j$; and $T_j^{+}(z_j(x_0)) < T_j^{+}(z_j(y))$, $T_j^{-}(z_j(y)) < T_j^{-}(z_j(x_0))$ for at least one j.

Definition 3 (Interactive fuzzy satisfying method, Sakawa 1987).

A decision plan $x_0 \in X$ is said to be a Pareto-optimal solution to the MOLPP (1) if there does not exist another $y \in X$ such that $z_i(y)) \le z_i(x_0)$ for all i, i≠j and $z_j(y) < z_j(x_0)$ for at least one j.

3 Obtaining I-Pareto-optimal solution and Pareto-optimal solution

Within the scope of multi-objective decision making theory, Pareto-optimality of solution is a necessary condition in order to guarantee the rationality of a decision. Now applying the concept of IF sets; Angelov (1997) formulated IF optimization (IFO) problem as follows

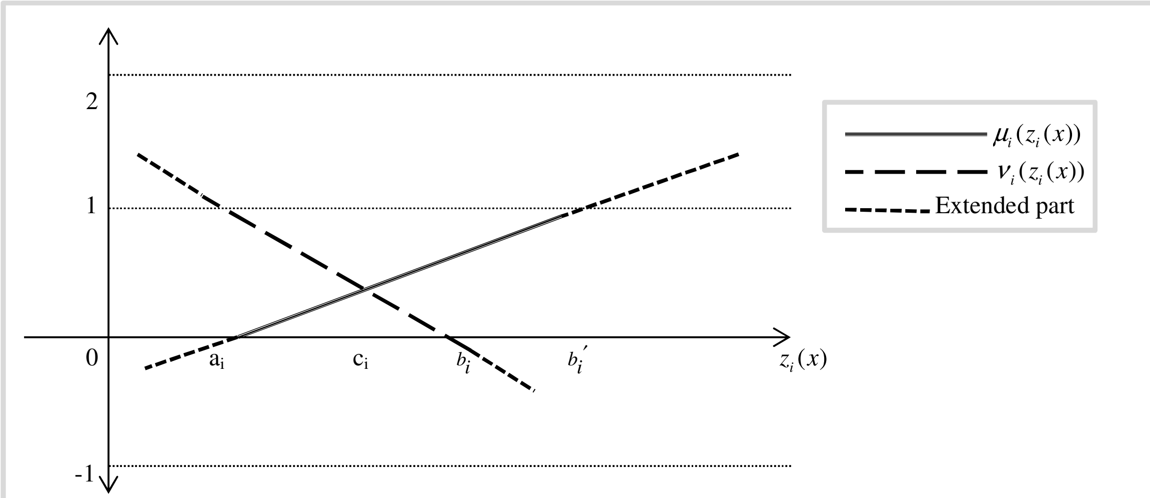

Fig. 1 Strictly monotonically extended membership function and non-membership function (not in scale)

$$
\begin{aligned}
&\max\ \mu_i(z_i(x)),\ \mathrm{i} = 1\ldots\mathrm{k}, \\
&\min\ \nu_i(z_i(x)),\ \mathrm{i} = 1\ldots\mathrm{k}, \\
&\text{subject to} \\
&\mu_i(z_i(x)) + \nu_i(z_i(x)) \le 1, \mathrm{i} = 1\ldots\mathrm{k}, \\
&\mu_i(z_i(x)) \ge \nu_i(z_i(x)), \mathrm{i} = 1\ldots\mathrm{k}, \\
&\nu_i(z_i(x)) \ge 0, \mathrm{i} = 1\ldots\mathrm{k}, \\
&\mathrm{x} \in \mathrm{X}.
\end{aligned}
\tag{3}
$$

Which was further simplified by Angelov (1997) as follows

$$
\begin{aligned}
&\max\ \alpha - \beta \\
&\text{subject to} \\
&\mu_i(z_i(x)) \ge \alpha, \mathrm{i} = 1\ldots\mathrm{k}, \\
&\nu_i(z_i(x)) \le \beta, \mathrm{i} = 1\ldots\mathrm{k}, \\
&\alpha \ge \beta, \\
&\alpha + \beta \le 1, \\
&\beta \ge 0, x \in X.
\end{aligned}
\tag{4}
$$

where α denotes the minimal degree of acceptance and β denotes the maximal degree of rejection of IF objective(s) and constraint(s). Without loss of generality, it may be assumed that all constraints of model (4) are active. It is observed that in problem formulations as well as during computations, strictly monotonic parts of definitions of membership function and non-membership function are used. And constraints $\mu_i(z_i(x)) \ge \alpha, \mathrm{i} = 1\ldots\mathrm{k}$, and $\nu_i(z_i(x)) \le \beta, \mathrm{i} = 1\ldots\mathrm{k}$, are suitably defined by relations $\alpha + \beta \le 1, \alpha \ge \beta$, $\beta \ge 0$ and hence $\alpha \ge 0$ in model (4). It is well known that a triangular IF goal to minimization type of objective function is usually denoted by $(a; b, b')$ where $\mu(z(a)) = 1$, $\nu(z(a)) = 0$, $\mu(z(b)) = 0$ and $\nu(z(b')) = 1$ with $b < b'$ as shown in Fig. 1.

Here the constraint $\alpha + \beta \le 1$ represents that sum of minimal degree of acceptance and maximal degree of rejection cannot exceed unity. But lemma 1 shows that on left side of a_i, strictly monotonic part of definitions results in sum of membership and non-membership values to always exceed unity. Hence in presence of constraint $\alpha + \beta \le 1$, objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k}$, may not attain value less than a_i. But since objective

function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ is of minimization type and IF goals and/or tolerances are imprecise in nature and any value less than a_i to objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ is more acceptable to DM, so presence of constraint $\alpha + \beta \le 1$ may result in less acceptable optimal solutions, as shown by numerical example 5.1.1. in this paper. Hence it is suggested to remove the constraint $\alpha + \beta \le 1$ so that objective functions $z_i(x), \mathrm{i} = 1\ldots\mathrm{k}$ can attain more preferable values.

Lemma 1 *Let* $(a_i; b_i, b_i')$ *be triangular IF goal of objective function* $z_i(x), \mathrm{i} = 1\ldots\mathrm{k}$ *of minimization type to MOLPP (1). Then for any* $a_i'' < a_i$, *usage of strictly monotonic part of definition of membership and non-membership function during computation yields that sum of membership degree and non-membership degree exceeds unity i.e.* $\mu_i(z_i(a_i'')) + \nu_i(z_i(a_i'')) > 1, \forall a_i'' < a_i$.

Proof Suppose $(a_i; b_i, b_i')$ be triangular IF goal to objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ of minimization type to MOLPP (1) under IF environment. Then $\forall i = 1\ldots k,$ from definition, it is known that $\mu_i(z_i(a_i)) = 1$ and $\nu_i(z_i(a_i)) = 0$ and $\mu_i(z_i(b_i)) = 0$ and $\nu_i(z_i(b_i')) = 1$ with $b_i < b_i'$. For all $i = 1\ldots k$, let us choose any $a_i'' < a_i$ such that $\forall i = 1\ldots k, a_i'' = z_i(x)$ for some $\mathrm{x} \in \mathrm{X}$. Then strictly monotonic property of membership function gives $\mu_i(z_i(a_i'')) = \dfrac{b_i - a_i''}{b_i - a_i}$ and non-membership function gives $\nu_i(z_i(a_i'')) = -\dfrac{a_i - a_i''}{b_i' - a_i}$.

Now $\forall i = 1\ldots k,$ we have

$$\mu_i(z_i(a_i'')) + \nu_i(z_i(a_i'')) = \frac{b_i - a_i''}{b_i - a_i} - \frac{a_i - a_i''}{b_i' - a_i} = 1 + \frac{(b_i' - b_i)(a_i - a_i'')}{(b_i - a_i)(b_i' - a_i)} > 1 \;\; \forall i = 1\ldots k.$$

Hence the lemma■

From Fig. 1 as well as lemma 1, it is clear that on extended part of membership and non-membership function (as represented by dotted line on left side of ai) sum of membership and non-membership values always exceed unity.

And classical definition of membership functions of IF set restricts upper limit at unity. But if constraint $\alpha \le 1$ be added to the model, it does not allow objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ to attain values less than a_i. But since IF goals and/or tolerances are imprecise in nature and objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ is of minimization type, DM will be happier to have optimal value of $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ as less than a_i as possible. So constraint $\alpha \le 1$ may result in less preferable values as optimal solutions to each IF objective function, as shown by numerical example 5.1.2. in this paper. Hence it is suggested not to add constraint $\alpha \le 1$ so that objective functions $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ may attain values as less than a_i as possible.

On the other hand, the constraint $\alpha \ge \beta$ suggests that minimal degree of acceptance cannot be less than maximum degree of rejection. This constraint further implies that objective function $z_i(x)$ may not attain values greater than c_i (c_i being abscissa of the point of intersection of membership function and non-membership function of objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k}$). But since IF goals and/or tolerances are imprecise in nature, in many cases such constraint may make the problem infeasible, as shown by numerical example 5.2.1. Hence it is suggested to remove the constraint $\alpha \ge \beta$ so that optimal solution may exist with objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ attaining values greater than c_i.

And constraint $\alpha \ge 0$ does not allow any objective function $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ to take value greater than b_i. But since IF goals and/or tolerances are imprecise in nature, in many cases such constraint may make the problem infeasible, as shown by numerical example 5.2.2. in this paper. Hence it is suggested to remove constraint $\alpha \ge 0$ so that optimal solution may exist and objective function(s) $z_i(x), \mathrm{i} = 1\ldots\mathrm{k},$ may attain

value(s) greater than b_i. It is analogous to removal of upper limit of membership function of fuzzy set theory as suggested by Wu et al (2015). Therefore classical membership function now transforms into one strictly monotonic function, which seems more useful in decision making under IF environment.

And classical definition of non-membership functions of IF set restricts upper limit at unity. If such constraint $\beta \le 1$ be added to the problem, it may lead to an infeasible solution, as shown by numerical example 5.2.3. Hence it is suggested not to add constraint $\beta \le 1$. Similarly removal of constraint $\beta \ge 0$ is suggested. Therefore classical non-membership function now transforms into one strictly monotonic function, which seems more useful in decision making under IF environment.

Hence in an MOLPP under IF environment, it is suggested to use strictly monotonic $T^{(+)}$-characteristic functions in place of membership functions and strictly monotonic $T^{(-)}$-characteristic functions in place of non-membership functions; and hence model (3) may be rewritten as follows

$$\begin{aligned} &\max\ \alpha - \beta \qquad (5)\\ &\text{subject to}\\ &T_i^{m}(z_i(x)) \ge \alpha, \text{i} = 1\ldots \text{k},\\ &T_i^{n}(z_i(x)) \le \beta, \text{i} = 1\ldots \text{k},\\ &x \in X, \alpha, \beta \text{ unrestricted in sign.}\end{aligned}$$

The task is to find an I-Pareto-optimal solution to model (5) and hence Pareto-optimal solution to MOLPP (1) under IF environment. To find I-Pareto-optimal solution to problem (5), the following lemmas are useful.

Lemma 2 *If $x^* \in X$ is unique optimal solution to model (5), then x^* is Pareto-optimal solution to the MOLPP (1) under IF environment.*

Proof Let $x^* \in X$ be unique I-Pareto-optimal solution to model (5). If possible let x^* be not Pareto-optimal solution to model (1) under IF environment. Then from definition of Pareto-optimality, there exist at least one $y \in X$ such that $z_i(y)) \le z_i(x^*)$ for all i, i $\ne$ j and $z_j(y) < z_j(x^*)$ for at least one j. Since each $T^{(+)}$-characteristic function $T_i^{(+)}(z_i)$ is strictly monotonic decreasing for minimization type of objective function $z_i \forall i = 1..k$ and each $T^{(-)}$-characteristic function $T_i^{(-)}(z_i)$ is strictly monotonic increasing function for minimizing type of objective function z_i, i = 1…k, we have $T_i^{(+)}(z_i(x^*)) \le T_i^{(+)}(z_i(y))\ \forall$ i =1…k, i $\ne j$ and $T_j^{(+)}(z_j(x^*)) < T_j^{(+)}(z_j(y))$ and $T_i^{(-)}(z_i(x^*)) \ge T_i^{(-)}(z_i(y))\ \forall$ i =1…k, i $\ne j$ and $T_j^{(-)}(z_j(x^*)) > T_j^{(-)}(z_j(y))$. It contradicts that x^* is unique optimal solution to model (5). Therefore the assumption is wrong. Hence x^* is Pareto-optimal solution to MOLPP (1) under IF environment■

Lemma 3 *If $x^* \in X$ is a Pareto-optimal solution of the MOLPP (1) under IF environment, then x^* is an I-Pareto-optimal solution of the problem (5) for some $< T_i^{(+)}, T_i^{(-)} >$, i=1…k.*

Proof Let $x^* \in X$ be Pareto-optimal solution to MOLPP (1) under IF environment. If possible let x^* be not an I-Pareto optimum solution to model (5) for any pair of $T^{(+)}$-characteristic function and $T^{(-)}$-characteristic function $< T_i^{(+)}, T_i^{(-)} >$, i = 1…k to objective function z_i, i = 1…k. Then for given functions $T_i^{(+)}, T_i^{(-)}$, i = 1…k, there exists $x \in X$ such that $T_j^{(+)}(z_j(x)) > T_j^{(+)}(z_j(x^*))$ for some j and $T_i^{(+)}(z_i(x)) \ge T_i^{(+)}(z_i(x^*)), i = 1\ldots k, i \ne j$ and $T_j^{(-)}(z_j(x)) < T_j^{(-)}(z_j(x^*))$ for some j and $T_i^{(-)}(z_i(x)) \le T_i^{(-)}(z_i(x^*)),\ i = 1\ldots k, i \ne j$. Since $T_i^{(+)}$ is strictly decreasing function and $T_i^{(-)}$ is strictly increasing function for each i = 1…k, we conclude $z_j(x) < z_j(x^*)$ for some j and $z_i(x) \le z_i(x^*)$, $i = 1\ldots k, i \ne j$. It contradicts that x^* is Pareto-optimal solution to the MOLPP (1) under IF environment. Therefore the assumption is wrong. Hence x* is an optimal solution to model (5) for pair of functions $< T_i^{(+)}, T_i^{(-)} >$, i = 1…k■

From lemma 2 and lemma 3, it is clear that if the uniqueness of the optimal solution x^* to model (5) is not guaranteed, it feels necessary to perform Pareto-optimality test for x^*.

And Pareto-optimality test for x^* can be performed by solving optimization problem with decision variables $x=(x_1,x_2,...,x_n)^T$, $\Omega=(\Omega_1,\Omega_2,...,\Omega_k)^T$ and $\mho=(\mho_1,\mho_2,...,\mho_k)^T$ as follows

$$\max \sum_{i=1}^{k}\Omega_i+\sum_{i=1}^{k}\mho_i \tag{6}$$

$$\begin{aligned}&\text{subject to}\\ &T_i^m(z_i(x))-\Omega_i\ge T_i^m(z_i(x^*)), i=1\ldots k,\\ &T_i^n(z_i(x))+\mho_i\le T_i^n(z_i(x^*)), i=1\ldots k,\\ &x\in X, \Omega_i, \mho_i\ge 0, i=1\ldots k.\end{aligned}$$

Lemma 4 *Let* $\bar{x},\bar{\Omega}$ and $\bar{\mho}$ *are optimal solutions to model (6). Then*

(1) If $\bar{\Omega}_i=0=\bar{\mho}_i, \forall i=1\ldots k$, *then x* is Pareto-optimal solution of the MOLPP (1) under IF environment.*

(2) If at least one $\bar{\Omega}_i>0$ *or* $\bar{\mho}_i>0$, *then x* is not Pareto-optimal solution to MOLPP under IF environment (1). Instead of x*,* $\bar{x}$ *is Pareto-optimal solution to the MOLPP under IF environment (1).*

Proof of analogous lemma is recorded in literature. Also other Pareto-optimality tests exist and may be used in place of proposed model (6).

4 General method to solve multi-objective programming problem under IF environment

The above ideas can be further integrated into a general framework and an algorithm may be developed to obtain I-Pareto-optimal solution and hence Pareto-optimal solution to MOLPP (1) under IF environment. The steps of the proposed algorithm are synthesized as follows

Step 1. Obtain suitable $T^{(+)}$-characteristic function for each IF objective in such a way that higher value of $T^{(+)}$-characteristic function always gives better result for objective function. And simultaneously obtain suitable $T^{(-)}$-characteristic function for each IF objective in such a way that lower value of $T^{(-)}$-characteristic function gives better result for objective function. Well-defined $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions are always finite.
Step 2. Construct single objective optimization problem as model (5).
Step 3. Solve model (5) and let x^* be I-Pareto-optimal solution.
Step 4. To test Pareto-optimality to this I-Pareto-optimal solution, solve model (6).
Step 5. Let $\bar{x},\bar{\Omega}$ and $\bar{\mho}$ are optimal solutions of model (3) in step 4. Then two cases may arise:

(1) If $\bar{\Omega}_i=0=\bar{\mho}_i, \forall i=1\ldots k$, then x* is Pareto-optimal solution of the MOLPP (1) under IF environment.

(2) If at least one $\bar{\Omega}_i>0$ or $\bar{\mho}_i>0$, then x* is not Pareto-optimal solution to MOLPP (1) under IF environment. Instead of x*, $\bar{x}$ is Pareto-optimal solution to the MOLPP (1) under IF environment.

Step 6. This solution is I-Pareto-optimal as well as Pareto-optimal solution. The algorithm is complete.

5 Numerical examples

5.1. Numerical Example 1

Consider MOLPP under IF environment as follows

Table 1 Individual maximum and minimum of objective functions

Objective function	Individual maximum	Individual minimum
$z_1(x) = 5x_1 + 5x_2$	10	0
$z_2(x) = 3x_1 - 8.2x_2$	3.33	-14.06

Table 2 Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for	
		Membership functions	non-membership functions
$z_1(x)$	8	1.5	2
$z_2(x)$	-2	2	2.5

$$\begin{aligned} &\textit{imprecise} \;\; \max \; z_1(x) = 5x_1 + 5x_2 \\ &\textit{imprecise} \;\; \max \; z_2(x) = 3x_1 - 8.2x_2 \\ &\textit{subject to} \\ &5x_1 + 7x_2 \le 12, 9x_1 + x_2 \le 10, -5x_1 + 3x_2 \le 3, x_1,\; x_2 \ge 0. \end{aligned} \tag{7}$$

To construct classical membership and non-membership functions for each IF objective function $z_i(x), i = 1,2$, individual maximum and minimum of each IF objective function are computed and given in Table 1.

5.1.1. Better result on removal of constraint $\alpha + \beta \le 1$

Now suppose DM specifies goals, tolerances of IF objective functions as in Table 2. Based on these assigned goals and tolerances, membership and non-membership functions of each IF objective function may be formulated as follows

$$\mu_1(z_1(x)) = \begin{cases} 1 & , \textit{if } 5x_1 + 5x_2 \ge 8 \\ \dfrac{5x_1 + 5x_2 - 6.5}{1.5}, & \textit{if } \; 6.5 \le 5x_1 + 5x_2 \le 8 \\ 0 & , \textit{if } \; 5x_1 + 5x_2 \le 6.5 \end{cases} \text{and } \nu_1(z_1(x)) = \begin{cases} 1 & , \textit{if } 5x_1 + 5x_2 \le 6 \\ \dfrac{8 - (5x_1 + 5x_2)}{2}, & \textit{if } \; 6 \le 5x_1 + 5x_2 \le 8 \\ 0 & , \textit{if } \; 5x_1 + 5x_2 \ge 8 \end{cases}$$

$$\mu_2(z_2(x)) = \begin{cases} 1 & , \textit{if } 3x_1 - 8.2x_2 \ge -2 \\ \dfrac{3x_1 - 8.2x_2 + 4}{2}, & \textit{if } -4 \le 3x_1 - 8.2x_2 \le -2 \\ 0 & , \textit{if } \; 3x_1 - 8.2x_2 \le -4 \end{cases} \text{and } \nu_2(z_2(x)) = \begin{cases} 1 & , \textit{if } 3x_1 - 8.2x_2 \le -4.5 \\ \dfrac{-2 - (3x_1 - 8.2x_2)}{2.5}, & \textit{if } -4.5 \le 3x_1 - 8.2x_2 \le -2 \\ 0 & , \textit{if } \; 3x_1 - 8.2x_2 \ge -2 \end{cases}$$

Using these membership and non-membership functions, MOLPP (7) under IF environment may be converted into single objective optimization model as follows [see, e.g. Angelov 1997]

$$\begin{aligned} &\max \alpha - \beta \\ &\textit{subject to} \\ &\frac{5x_1 + 5x_2 - 6.5}{1.5} \ge \alpha, \;\; \frac{3x_1 - 8.2x_2 + 4}{2} \ge \alpha, \frac{8 - (5x_1 + 5x_2)}{2} \le \beta, \;\; \frac{-2 - (3x_1 - 8.2x_2)}{2.5} \le \beta, \\ &5x_1 + 7x_2 \le 12, \;\; 9x_1 + x_2 \le 10, \;\; -5x_1 + 3x_2 \le 3, \;\; \alpha \ge \beta, \;\; \alpha + \beta \le 1, \;\; \beta, x_1,\; x_2 \ge 0. \end{aligned}$$

Using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

Table 3 Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for	
		Membership functions	non-membership functions
$z_1(x)$	8	1.5	2
$z_2(x)$	-2	2	2.5

$$\alpha = 1,\ \beta = 0,\ x_1{}^* = 0.993,\ x_2{}^* = 0.607,\ z_1(x^*) = 8,\ z_2(x^*) = -2,$$
$$\mu_1(z_1(x^*)) = \mu_2(z_2(x^*)) = 1,\ \nu_1(z_1(x^*)) = \nu_2(z_2(x^*)) = 0.$$

As discussed in Sect. 3 of this paper, if the constraint $\alpha + \beta \le 1$ be removed, using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

$$\alpha = 1.131,\ \beta = 0,\ x_1{}^{**} = 1.045,\ x_2{}^{**} = 0.594,\ z_1(x^{**}) = 8.197,\ z_2(x^{**}) = -1.738,$$
$$\mu_1(z_1(x^{**})) = 1.131, \mu_2(z_2(x^{**})) = 1.131, \nu_1(z_1(x^{**})) = -0.098, \nu_2(z_2(x^{**})) = -0.105,$$

To test Pareto optimality of these I-Pareto optimal solutions, the following problem is now solved

$$\max \qquad \sum_{i=1}^{2} \Omega_i + \sum_{i=1}^{2} \mho_i$$

subject to

$$\frac{5x_1 + 5x_2 - 6.5}{1.5} - \Omega_1 \ge 1.1311476, \quad \frac{3x_1 - 8.2x_2 + 4}{2} - \Omega_2 \ge 1.1311476,$$
$$\frac{8 - \left(5x_1 + 5x_2\right)}{2} + \mho_1 \le -0.0983608, \quad \frac{-2 - \left(3x_1 - 8.2x_2\right)}{2.5} + \mho_2 \le -0.1049180,$$
$$5x_1 + 7x_2 \le 12,\ 9x_1 + x_2 \le 10,\ -5x_1 + 3x_2 \le 3,$$
$$x_1,\ x_2, \Omega_i, \mho_i \ge 0,\ \forall \mathrm{i}=1,2.$$

Using Lingo 15.0.32, Pareto optimal solutions $\bar{x}$, $\bar{\Omega}$ and $\bar{\mho}$ are obtained as follows

$$\bar{\Omega}_1 = \bar{\Omega}_2 = 0 = \bar{\mho}_1 = \bar{\mho}_2,\ \bar{x}_1 = 1.045,\ \bar{x}_2 = 0.594, z_1(\bar{x}) = 8.197,\ z_2(\bar{x}) = -1.738$$
$$\mu_1(z_1(\bar{x})) = \ \mu_2(z_2(\bar{x})) = \ 1.131,\ \nu_1(z_1(\bar{x})) = -0.098,\ \nu_2(z_2(\bar{x})) = \ -0.105.$$

Therefore it is observed that removal of constraint $\alpha + \beta \le 1$ generates more acceptable I-Pareto optimal solutions with both objective functions attaining better values having higher degrees of acceptances as well as lower degrees of rejections. Hence these solutions x^{**} , $z(x^{**})$ i.e. $\bar{x}$, $z(\bar{x})$ are more acceptable to DM and they have the additional property of being Pareto optimal.

5.1.2.Better result without adding constraint $\alpha \le 1$ (so constraint $\alpha + \beta \le 1$ is not present)

Again suppose that DM specifies goals and tolerances for each IF objective function as given in Table 3. Based on these assigned goals and tolerances, membership and non-membership functions of each of the imprecise objective functions may be formulated as follows

$$\mu_1\left(z_1(x)\right) = \begin{cases} 1 & , \text{if } 5x_1 + 5x_2 \ge 8 \\ \dfrac{5x_1 + 5x_2 - 6.5}{1.5} & , \text{ if } 6.5 \le 5x_1 + 5x_2 \le 8 \\ 0 & , \text{ if } 5x_1 + 5x_2 \le 6.5 \end{cases}$$

$$\nu_1\left(z_1(x)\right)=\begin{cases}1 & , if\, 5x_1+5x_2\le 6\\ \dfrac{8-\left(5x_1+5x_2\right)}{2} & , if\;\, 6\le 5x_1+5x_2\le 8\\ 0 & , if\;\, 5x_1+5x_2\ge 8\end{cases}$$

$$\mu_2\left(z_2(x)\right)=\begin{cases}1 & , if\, 3x_1-8.2x_2\ge -2\\ \dfrac{3x_1-8.2x_2+4}{2} & , if\, -4\le 3x_1-8.2x_2\le -2\\ 0 & , if\;\, 3x_1-8.2x_2\le -4\end{cases}$$

$$\nu_2\left(z_2(x)\right)=\begin{cases}1 & , if\, 3x_1-8.2x_2\le -4.5\\ \dfrac{-2-\left(3x_1-8.2x_2\right)}{2.5} & , if\, -4.5\le 3x_1-8.2x_2\le -2\\ 0 & , if\;\, 3x_1-8.2x_2\ge -2\end{cases}$$

Using these membership and non-membership functions, MOLPP under IF environment may be converted into single objective optimization model as follows [see, e.g. Angelov 1997]

$$\begin{aligned}&\max \quad \alpha-\beta\\ &\textit{subject to}\\ &\frac{5x_1+5x_2-6.5}{1.5}\ge\alpha,\;\; \frac{3x_1-8.2x_2+4}{2}\ge\alpha,\\ &\frac{8-\left(5x_1+5x_2\right)}{2}\le\beta,\;\; \frac{-2-\left(3x_1-8.2x_2\right)}{2.5}\le\beta,\\ &5x_1+7x_2\le 12,\;\; 9x_1+x_2\le 10,\;\; -5x_1+3x_2\le 3,\\ &\alpha\ge\beta,\;\; \alpha\le 1,\, x_1,\;\, x_2\ge 0.\end{aligned}$$

Using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

$$\begin{aligned}&\alpha=1,\;\beta=0,\;x_1^*=0.993,\;x_2^*=0.607,\;z_1(x^*)=8,\;z_2(x^*)=-2,\\ &\mu_1(z_1(x^*))=\mu_2(z_2(x^*))=1,\nu_1(z_1(x^*))=\nu_2(z_2(x^*))=0.\end{aligned}$$

As discussed in Sect. 3 of this paper, if constraint $\alpha\le 1$ be removed, using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

$$\begin{aligned}&\alpha=1.131,\;\beta=0,\;x_1{**}=1.045,\;x_2{**}=0.594,\;z_1(x{**})=8.197,\;z_2(x{**})=-1.738,\\ &\mu_1(z_1(x{**}))=1.131,\mu_2(z_2(x{**}))=1.131,\nu_1(z_1(x{**}))=-0.098,\nu_2(z_2(x{**}))=-0.105.\end{aligned}$$

To test Pareto optimality of these I-Pareto optimal solutions, following problem is solved

Table 4 Individual maximum and minimum of objective functions

Objective function	Individual maximum	Individual minimum
$z_1(x) = 5x_1 + 5x_2$	10	0
$z_2(x) = 5x_1 + x_2$	6	0
$z_3(x) = 3x_1 - 8.2x_2$	3.33	-14.06

Table 5 Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for	
		Membership functions	non-membership functions
$z_1(x)$	7	1.5	2
$z_2(x)$	2	2	2.5
$z_3(x)$	-2	2	2.5

$$\max \quad \sum_{i=1}^{2} \Omega_i + \sum_{i=1}^{2} \mho_i$$

subject to

$$\frac{5x_1 + 5x_2 - 6.5}{1.5} - \Omega_1 \geq 1.1311476, \quad \frac{3x_1 - 8.2x_2 + 4}{2} - \Omega_2 \geq 1.1311476,$$

$$\frac{8 - \left(5x_1 + 5x_2\right)}{2} + \mho_1 \leq -0.0983608, \quad \frac{-2 - \left(3x_1 - 8.2x_2\right)}{2.5} + \mho_2 \leq -0.1049180,$$

$$5x_1 + 7x_2 \leq 12, \;\; 9x_1 + x_2 \leq 10, \;\; -5x_1 + 3x_2 \leq 3, x_1, \; x_2, \Omega_i, \mho_i \geq 0, \;\; \forall \text{i=1,2.}$$

Using Lingo 15.0.32, Pareto optimal solutions $\bar{x}$, $\bar{\Omega}$ and $\bar{\mho}$ are obtained as follows

$$\bar{\Omega}_1 = \bar{\Omega}_2 = 0 = \bar{\mho}_1 = \bar{\mho}_2, \;\; \bar{x}_1 = 1.045, \; \bar{x}_2 = 0.594, \; z_1(\bar{x}) = 8.197, \; z_2(\bar{x}) = -1.738$$
$$\mu_1(z_1(\bar{x})) = \; \mu_2(z_2(\bar{x})) = \; 1.131, \;\; \nu_1(z_1(\bar{x})) = -0.098, \;\; \nu_2(z_2(\bar{x})) = \;\; -0.105.$$

Therefore it is observed that removal of constraint $\alpha \leq 1$ generates more acceptable I-Pareto optimal solution to MOLPP (7) under IF environment in which both objective functions have better values having higher degrees of acceptances as well as lower degrees of rejections. Hence these solutions x^{**} , $z(x^{**})$ i.e. $\bar{x}$, $z(\bar{x})$ are more acceptable to DM and they have the additional property of being Pareto optimal.

5.2. Numerical Example 2

Consider MOLPP under IF environment as follows

$$\textit{imprecise} \;\; \max 5x_1 + 5x_2, \textit{imprecise} \;\; \min \;\; 5x_1 + x_2, \textit{imprecise} \;\; \max \;\; 3x_1 - 8.2x_2 \tag{8}$$
$$\textit{subject to} \;\; 5x_1 + 7x_2 \leq 12, 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, x_1, \; x_2 \geq 0.$$

To construct classical membership and non-membership functions for each IF objective function $z_i(x), i = 1, 2, 3$, individual maximum and minimum of each IF objective function are computed and given in Table 4.

5.2.1. Solution found on removal of constraint $\alpha \geq \beta$

Now suppose that DM specifies goals, tolerances of imprecise objective functions as in Table 5.

Based on these assigned goals and tolerances, membership and non-membership functions of each of the IF objective functions are formulated as follows

$$\mu_1(z_1(x))=\begin{cases}1 & , if\ 5x_1+5x_2\geq 7\\ \dfrac{5x_1+5x_2-5.5}{1.5} & , if\ 5.5\leq 5x_1+5x_2\leq 7\\ 0 & , if\ 5x_1+5x_2\leq 5.5\end{cases} \text{ and } \nu_1(z_1(x))=\begin{cases}1 & , if\ 5x_1+5x_2\leq 5\\ \dfrac{7-(5x_1+5x_2)}{2} & , if\ 5\leq 5x_1+5x_2\leq 7\\ 0 & , if\ 5x_1+5x_2\geq 7\end{cases}$$

$$\mu_2(z_2(x))=\begin{cases}1 & , if\ 5x_1+x_2\leq 2\\ \dfrac{4-(5x_1+x_2)}{2} & , if\ 2\leq 5x_1+x_2\leq 4\\ 0 & , if\ 5x_1+x_2\geq 4\end{cases} \text{ and } \nu_2(z_2(x))=\begin{cases}1 & , if\ 5x_1+x_2\geq 4.5\\ \dfrac{(5x_1+x_2)-2}{2.5} & , if\ 2\leq 5x_1+x_2\leq 4.5\\ 0 & , if\ 5x_1+x_2\leq 2\end{cases}$$

$$\mu_3(z_3(x))=\begin{cases}1 & , if\ 3x_1-8.2x_2\geq -2\\ \dfrac{3x_1-8.2x_2+4}{2} & , if\ -4\leq 3x_1-8.2x_2\leq -2\\ 0 & , if\ 3x_1-8.2x_2\leq -4\end{cases} \text{ and } \nu_3(z_3(x))=\begin{cases}1 & , if\ 3x_1-8.2x_2\leq -4.5\\ \dfrac{-2-(3x_1-8.2x_2)}{2.5} & , if\ -4.5\leq 3x_1-8.2x_2\leq -2\\ 0 & , if\ 3x_1-8.2x_2\geq -2\end{cases}$$

Using these membership and non-membership functions, MOLPP (8) under IF environment may be converted into one single objective optimization problem as follows [see, e.g. Angelov 1997]

$$\max\ \alpha-\beta$$

subject to

$$\frac{5x_1+5x_2-5.5}{1.5}\geq\alpha,\quad \frac{4-(5x_1+x_2)}{2}\geq\alpha,\quad \frac{3x_1-8.2x_2+4}{2}\geq\alpha,$$

$$\frac{7-(5x_1+5x_2)}{2}\leq\beta,\quad \frac{(5x_1+x_2)-2}{2.5}\leq\beta,\quad \frac{-2-(3x_1-8.2x_2)}{2.5}\leq\beta,$$

$$5x_1+7x_2\leq 12,\ 9x_1+x_2\leq 10,\ -5x_1+3x_2\leq 3, \alpha\geq\beta,\ \alpha+\beta\leq 1, \beta, x_1,\ x_2\geq 0.$$

Using Lingo 15.0.32, it is found that the *problem has no feasible solution*. As discussed in Sect. 3 of this paper, if the constraint that $\alpha\geq\beta$ be removed, using Lingo 15.0.32, I-Pareto optimal solutions may be obtained as follows

$$\alpha=0.284,\ \beta=0.572,\ x_1^*=0.561,\ x_2^*=0.624, z_1(x^*)=5.927,\ z_2(x^*)=3.431,\ z_3(x^*)=-3.431,$$

$$\mu_1(z_1(x^*))=\mu_2(z_2(x^*))=\mu_3(z_3(x^*))=0.284, \nu_1(z_1(x^*))=0.537, \nu_2(z_2(x^*))=\nu_3(z_3(x^*))=0.572.$$

To test the Pareto optimality of these I-Pareto optimal solutions, the following problem is solved

$$\max \sum_{i=1}^{3}\Omega_i+\sum_{i=1}^{3}\mho_i$$

subject to

$$\frac{5x_1+5x_2-5.5}{1.5}-\Omega_1\geq 0.2844037,\quad \frac{4-(5x_1+x_2)}{2}-\Omega_2\geq 0.2844037,\quad \frac{3x_1-8.2x_2+4}{2}-\Omega_3\geq 0.2844037,$$

$$\frac{7-(5x_1+5x_2)}{2}+\mho_1\leq 0.5366972,\quad \frac{(5x_1+x_2)-2}{2.5}+\mho_2\leq 0.5724771,\quad \frac{-2-(3x_1-8.2x_2)}{2.5}+\mho_3\leq 0.5724771,$$

$$5x_1+7x_2\leq 12,\ 9x_1+x_2\leq 10,\ -5x_1+3x_2\leq 3, x_1,\ x_2, \Omega_i, \mho_i\geq 0,\ \forall i=1,2,3.$$

Table 6 Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for Membership functions	Tolerances for non-membership functions
$z_1(x)$	8	1.5	2
$z_2(x)$	1.5	2	2.5
$z_3(x)$	-2	2	2.5

Using Lingo 15.0.32, Pareto optimal solutions $\bar{x}$, $\bar{\Omega}$ and $\bar{\mho}$ are obtained as follows

$$\bar{\Omega}_1=\bar{\Omega}_2=\bar{\Omega}_3=0=\bar{\mho}_1=\bar{\mho}_2=\bar{\mho}_3,\ \bar{x}_1=0.561,\ \bar{x}_2=0.624,\ z_1(\bar{x})=5.927,\ z_2(\bar{x})=3.431,\ z_3(\bar{x})=-3.431,$$
$$\mu_1(z_1(\bar{x}))=\mu_2(z_2(\bar{x}))=\mu_3(z_3(\bar{x}))=0.284,\ \nu_1(z_1(\bar{x}))=0.537,\ \nu_2(z_2(\bar{x}))=\nu_3(z_3(\bar{x}))=0.572.$$

Therefore on removal of constraint $\alpha \ge \beta$, it is found that MOLPP (8) has Pareto optimal solutions under IF environment specified by DM whereas presence of the constraint $\alpha \ge \beta$ made problem infeasible.

5.2.2. Solution found on removal of constraint $\alpha \ge 0$ (so constraint $\alpha \ge \beta$ is not present)

Again suppose that DM specifies goals, tolerances of each IF objective function as in Table 6. Based on these assigned goals, and tolerances, membership and non-membership functions of each IF objective function are formulated as follows

$$\mu_1(z_1(x))=\begin{cases}1 & ,\ if\ 5x_1+5x_2\ge 8\\ \dfrac{5x_1+5x_2-6.5}{1.5} & ,\ if\ 6.5\le 5x_1+5x_2\le 8\\ 0 & ,\ if\ 5x_1+5x_2\le 6.5\end{cases} \text{ and } \nu_1(z_1(x))=\begin{cases}1 & ,\ if\ 5x_1+5x_2\le 6\\ \dfrac{8-(5x_1+5x_2)}{2} & ,\ if\ 6\le 5x_1+5x_2\le 8\\ 0 & ,\ if\ 5x_1+5x_2\ge 8\end{cases}$$

$$\mu_2(z_2(x))=\begin{cases}1 & ,\ if\ 5x_1+x_2\le 1.5\\ \dfrac{3.5-(5x_1+x_2)}{2} & ,\ if\ 1.5\le 5x_1+x_2\le 3.5\\ 0 & ,\ if\ 5x_1+x_2\ge 3.5\end{cases} \text{ and } \nu_2(z_2(x))=\begin{cases}1 & ,\ if\ 5x_1+x_2\ge 4\\ \dfrac{(5x_1+x_2)-1.5}{2.5} & ,\ if\ 1.5\le 5x_1+x_2\le 4\\ 0 & ,\ if\ 5x_1+x_2\le 1.5\end{cases}$$

$$\mu_3(z_3(x))=\begin{cases}1 & ,\ if\ 3x_1-8.2x_2\ge -2\\ \dfrac{3x_1-8.2x_2+4}{2} & ,\ if\ -4\le 3x_1-8.2x_2\le -2\\ 0 & ,\ if\ 3x_1-8.2x_2\le -4\end{cases} \text{ and } \nu_3(z_3(x))=\begin{cases}1 & ,\ if\ 3x_1-8.2x_2\le -4.5\\ \dfrac{-2-(3x_1-8.2x_2)}{2.5} & ,\ if\ -4.5\le 3x_1-8.2x_2\le -2\\ 0 & ,\ if\ 3x_1-8.2x_2\ge -2\end{cases}$$

Using those membership and non-membership functions, MOLPP (8) under IF environment may be converted into single objective optimization problem as follows [see, e.g. Angelov 1997]

$$\begin{aligned}&\max \quad \alpha-\beta\\ &\textit{subject to}\\ &\frac{5x_1+5x_2-6.5}{1.5}\ge\alpha,\quad \frac{3.5-(5x_1+x_2)}{2}\ge\alpha,\quad \frac{3x_1-8.2x_2+4}{2}\ge\alpha,\\ &\frac{8-(5x_1+5x_2)}{2}\le\beta,\quad \frac{(5x_1+x_2)-1.5}{2.5}\le\beta,\quad \frac{-2-(3x_1-8.2x_2)}{2.5}\le\beta,\\ &5x_1+7x_2\le 12,\ 9x_1+x_2\le 10,\ -5x_1+3x_2\le 3, \alpha\ge\beta, \alpha+\beta\le 1, \beta, x_1,\ x_2\ge 0.\end{aligned}$$

Table 7 Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for	
		Membership functions	non-membership functions
$z_1(x)$	9.5	1.5	2
$z_2(x)$	1.5	2	2.5
$z_3(x)$	-2	2	2.5

Using Lingo 15.0.32, the problem has no feasible solution. If constraint $\alpha \geq 0$ be removed, problem becomes

$$\max \quad \alpha - \beta$$

subject to

$$\frac{5x_1 + 5x_2 - 6.5}{1.5} \geq \alpha, \quad \frac{3.5 - (5x_1 + x_2)}{2} \geq \alpha, \quad \frac{3x_1 - 8.2x_2 + 4}{2} \geq \alpha,$$

$$\frac{8 - (5x_1 + 5x_2)}{2} \leq \beta, \quad \frac{(5x_1 + x_2) - 1.5}{2.5} \leq \beta, \quad \frac{-2 - (3x_1 - 8.2x_2)}{2.5} \leq \beta,$$

$$5x_1 + 7x_2 \leq 12, \; 9x_1 + x_2 \leq 10, \; -5x_1 + 3x_2 \leq 3, \alpha + \beta \leq 1, \beta, x_1, \; x_2 \geq 0, \alpha \text{ unrestricted in sign.}$$

Using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

$$\alpha = -0.046, \; \beta = 0.837, \; x_1{}^* = 0.576, \; x_2{}^* = 0.710, \; z_1(x^*) = 6.43, \; z_2(x^*) = 3.59, \; z_3(x^*) = -4.094,$$
$$\mu_1(z_1(x^*)) = \mu_2(z_2(x^*)) = \mu_3(z_3(x^*)) = -0.046, \nu_1(z_1(x^*)) = 0.785, \; \nu_2(z_2(x^*)) = \nu_3(z_3(x^*)) = 0.837.$$

To test the Pareto optimality of these I-Pareto optimal solutions, following problem is solved

$$\max \sum_{i=1}^{3} \Omega_i + \sum_{i=1}^{3} \mho_i$$

subject to

$$\frac{5x_1 + 5x_2 - 6.5}{1.5} - \Omega_1 \geq -0.0458716, \quad \frac{8 - (5x_1 + 5x_2)}{2} + \mho_1 \leq 0.7844037,$$

$$\frac{3.5 - (5x_1 + x_2)}{2} - \Omega_2 \geq -0.0458716, \quad \frac{(5x_1 + x_2) - 1.5}{2.5} + \mho_2 \leq 0.8366972,$$

$$\frac{3x_1 - 8.2x_2 + 4}{2} - \Omega_3 \geq -0.0458716, \quad \frac{-2 - (3x_1 - 8.2x_2)}{2.5} + \mho_3 \leq 0.8366972,$$

$$5x_1 + 7x_2 \leq 12, \; 9x_1 + x_2 \leq 10, \; -5x_1 + 3x_2 \leq 3, x_1, \; x_2, \Omega_i, \mho_i \geq 0 \;\; \forall i=1, 2, 3.$$

Using Lingo 15.0.32, Pareto optimal solutions $\overline{x}$, $\overline{\Omega}$ and $\overline{\mho}$ are obtained as follows

$$\overline{\Omega}_1 = \overline{\Omega}_2 = \overline{\Omega}_3 = 0 = \overline{\mho}_1 = \overline{\mho}_2 = \overline{\mho}_3, \; \overline{x}_1 = 0.576, \; \overline{x}_2 = 0.710, \; z_1(\overline{x}) = 6.43, \; z_2(\overline{x}) = 3.59, \; z_3(\overline{x}) = -4.094,$$
$$\mu_1(z_1(\overline{x})) = \mu_2(z_2(\overline{x})) = \mu_3(z_3(\overline{x})) = -0.046, \; \nu_1(z_1(\overline{x})) = 0.785, \; \nu_2(z_2(\overline{x})) = \nu_3(z_3(\overline{x})) = 0.837.$$

Therefore after removal of constraint $\alpha \geq 0$, it is found that MOLPP (8) has Pareto optimal solutions under IF environment specified by DM whereas presence of the constraint $\alpha \geq 0$ made problem infeasible.

5.2.3. Solution found without adding constraint $\beta \leq 1$ (so constraints $\alpha \geq \beta$ and $\alpha \geq 0$ are not present)

Again suppose that DM specifies goals, tolerances of IF objective functions as in Table 7.

Based on these assigned goals and tolerances, membership and non-membership functions of each of the IF objectives are formulated as follows

$$\mu_1(z_1(x))=\begin{cases}1 & , if\ 5x_1+5x_2\geq 9.5\\ \dfrac{5x_1+5x_2-8}{1.5}, & if\ 8\leq 5x_1+5x_2\leq 9.5\\ 0 & , if\ 5x_1+5x_2\leq 8\end{cases},\quad \nu_1(z_1(x))=\begin{cases}1 & , if\ 5x_1+5x_2\leq 7.5\\ \dfrac{9.5-(5x_1+5x_2)}{2}, & if\ 7.5\leq 5x_1+5x_2\leq 9.5\\ 0 & , if\ 5x_1+5x_2\geq 9.5\end{cases}$$

$$\mu_2(z_2(x))=\begin{cases}1 & , if\ 5x_1+x_2\leq 1.5\\ \dfrac{3.5-(5x_1+x_2)}{2}, & if\ 1.5\leq 5x_1+x_2\leq 3.5\\ 0 & , if\ 5x_1+x_2\geq 3.5\end{cases},\quad \nu_2(z_2(x))=\begin{cases}1 & , if\ 5x_1+x_2\geq 4\\ \dfrac{(5x_1+x_2)-1.5}{2.5}, & if\ 1.5\leq 5x_1+x_2\leq 4\\ 0 & , if\ 5x_1+x_2\leq 1.5\end{cases}$$

$$\mu_3(z_3(x))=\begin{cases}1 & , if\ 3x_1-8.2x_2\geq -2\\ \dfrac{3x_1-8.2x_2+4}{2}, & if\ -4\leq 3x_1-8.2x_2\leq -2\\ 0 & , if\ 3x_1-8.2x_2\leq -4\end{cases},\quad \nu_3(z_3(x))=\begin{cases}1 & , if\ 3x_1-8.2x_2\leq -4.5\\ \dfrac{-2-(3x_1-8.2x_2)}{2.5}, & if\ -4.5\leq 3x_1-8.2x_2\leq -2\\ 0 & , if\ 3x_1-8.2x_2\geq -2\end{cases}$$

Using those membership and non-membership functions, MOLPP (8) under IF environment is converted into single objective optimization model as follows [see, e.g. Angelov 1997]

$$\begin{aligned}&\max \quad \alpha-\beta\\ &\textit{subject to}\\ &\frac{5x_1+5x_2-8}{1.5}\geq\alpha,\ \frac{3.5-(5x_1+x_2)}{2}\geq\alpha,\ \frac{3x_1-8.2x_2+4}{2}\geq\alpha,\\ &\frac{9.5-(5x_1+5x_2)}{2}\leq\beta,\ \frac{(5x_1+x_2)-1.5}{2.5}\leq\beta,\ \frac{-2-(3x_1-8.2x_2)}{2.5}\leq\beta,\\ &5x_1+7x_2\leq 12,\ 9x_1+x_2\leq 10,\ -5x_1+3x_2\leq 3,\\ &\beta\leq 1, \alpha+\beta\leq 1,\ \beta, x_1,\ x_2\geq 0, \alpha \text{ unrestricted in sign.}\end{aligned}$$

Using Lingo 15.0.32, it is found that the problem has no feasible solution.
As discussed in Sect. 3 of this paper, if the constraint $\beta\leq 1$ be not added, the problem becomes

$$\begin{aligned}&\max \quad \alpha-\beta\\ &\textit{subject to}\\ &\frac{5x_1+5x_2-8}{1.5}\geq\alpha,\ \frac{3.5-(5x_1+x_2)}{2}\geq\alpha,\ \frac{3x_1-8.2x_2+4}{2}\geq\alpha,\\ &\frac{9.5-(5x_1+5x_2)}{2}\leq\beta,\ \frac{(5x_1+x_2)-1.5}{2.5}\leq\beta,\ \frac{-2-(3x_1-8.2x_2)}{2.5}\leq\beta,\\ &5x_1+7x_2\leq 12,\ 9x_1+x_2\leq 10,\ -5x_1+3x_2\leq 3,\\ &\alpha+\beta\leq 1,\ \beta, x_1,\ x_2\geq 0, \alpha \text{ unrestricted in sign.}\end{aligned}$$

Using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

$$\alpha=-0.349,\ \beta=1.079, x_1^*=0.675,\ x_2^*=0.820,\ z_1(x^*)=7.477,\ z_2(x^*)=4.197, z_3(x^*)=-4.697.$$
$$\mu_1(z_1(x^*))=\mu_2(z_2(x^*))=\mu_3(z_3(x^*))=-0.349, \nu_1(z_1(x^*))=1.011,\ \nu_2(z_2(x^*))=\nu_3(z_3(x^*))=1.079.$$

To test the Pareto optimality of these I-Pareto optimal solutions, following problem is now solved

$$\max \sum_{i=1}^{3}\Omega_i+\sum_{i=1}^{3}\mho_i$$
subject to
$$\frac{5x_1+5x_2-8}{1.5}-\Omega_1\ge -0.3486239,\ \frac{9.5-\left(5x_1+5x_2\right)}{2}+\mho_1\le 1.011468,$$
$$\frac{3.5-\left(5x_1+x_2\right)}{2}-\Omega_2\ge -0.3486239,\ \frac{\left(5x_1+x_2\right)-1.5}{2.5}+\mho_2\le 1.0788999,$$
$$\frac{3x_1-8.2x_2+4}{2}-\Omega_3\ge -0.3486239, \frac{-2-\left(3x_1-8.2x_2\right)}{2.5}+\mho_3\le 1.0788999,$$
$$5x_1+7x_2\le 12,\ 9x_1+x_2\le 10,\ -5x_1+3x_2\le 3, x_1,\ x_2, \Omega_i, \mho_i\ge 0,\ \forall \text{i=1,2,3.}$$

Using Lingo 15.0.32, Pareto optimal solutions $\bar{x}$, $\bar{\Omega}$ and $\bar{\mho}$ are obtained as follows

$$\bar{\Omega}_1=\bar{\Omega}_2=\bar{\Omega}_3=0=\bar{\mho}_1=\bar{\mho}_2=\bar{\mho}_3,\ \ \bar{x}_1=0.675,\ \bar{x}_2=0.820, z_1(\bar{x})=7.477,\ z_2(\bar{x})=4.197,\ z_3(\bar{x})=-4.697,$$
$$\mu_1(z_1(\bar{x}))=\ \mu_2(z_2(\bar{x}))=\ \mu_3(z_3(\bar{x}))=-0.349,\ \nu_1(z_1(\bar{x}))=1.011,\ \nu_2(z_2(\bar{x}))=\ \nu_3(z_3(\bar{x}))=1.079.$$

Therefore if constraint $\beta\le 1$ be not added, it is found that MOLPP (8) has Pareto optimal solutions under IF environment as specified by DM whereas presence of the constraint $\beta\le 1$ may make problem infeasible.

5.3. Numerical application of proposed algorithm in large scale steel plant

5.3.1. Usage of T(+)-characteristic functions and T(-)-characteristic functions in steel-iron industry problem

This example is based on paper titled *A multi-objective model for purchasing of bulk raw materials of a large-scale integrated steel plant* by Zhen Gao, Lixin Tang (2003) published in *International Journal of Production Economics*. Instead of assigning crisp goals of the objective functions, IF goals are assigned to objective functions. Moreover in place of membership and non-membership functions, $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions are used in this MOLPP with IF goals. Here the problem is of purchasing raw materials of a large scale steel plant. In steel-iron industry, selection of appropriate items is the key to reduce production cost. Decision on quantities of raw materials using strong professional knowledge on steel iron metallurgy is another key objective. Selection of vendors to keep stability and quality of supply of raw materials is also key objective in steel-iron industry. Using two dimensional vectors x_{ij} denoting order quantity of j^{th} item of raw materials from i^{th} vendor, and assuming the model as single time phase model, the constraints under consideration may be taken as follows: purchasing budget constraint, production demand constraint, inventory capacity constraint, technological requirement constraint, vendor resource constraint. Suppose the problem is to have seven vendors and thirteen items that belong to four large kinds of bulk raw materials- ore, lump ore, pellet, coal [see Gao 2003]. And analogous to Gao (2003), suppose the MOLPP with IF goals is

$$\begin{aligned}
&\text{Imprecise min } z_1(x)=0.112x_{11}+0.127x_{31}+0.122x_{41}+0.115x_{51}+0.119x_{71}+0.0654x_{12}+0.0621x_{22}\\
&\qquad +0.0586x_{32}+0.0602x_{62}+0.195x_{33}+0.185x_{53}+0.09521x_{14}+0.0975x_{34}\\
&\text{Imprecise min } z_2(x)=0.1x_{11}+0.155x_{31}+0.17x_{41}+0.12x_{51}+0.2x_{71}+0.1x_{12}+0.25x_{22}+0.15x_{32}+0.3x_{62}\\
&\qquad +0.15x_{33}+0.12x_{53}+0.1x_{14}+0.15x_{34}\\
&\text{Imprecise min } z_3(x)=0.2x_{11}+0.1x_{31}+0.15x_{41}+0.17x_{51}+0.13x_{71}+0.2x_{12}+0.1x_{22}+0.15x_{32}+0.22x_{62}\\
&\qquad +0.15x_{33}+0.17x_{53}+0.2x_{14}+0.15x_{34}
\end{aligned} \tag{9}$$

Table 8 Purchasing decision of bulk raw materials of a large-scale integrated steel plant with IF goals

Cases		I			II			III		
Objective functions		$z_1(x)$	$z_2(x)$	$z_3(x)$	$z_1(x)$	$z_2(x)$	$z_3(x)$	$z_1(x)$	$z_2(x)$	$z_3(x)$
Goals		16	18.5	27.5	15.5	22.5	23.8	15.1	20.4	24.4
Tolerances of	$T_i^{(+)}(z_i(x))$	0.3	0.3	0.3	0.5	0.5	0.5	0.4	0.4	0.4
	$T_i^{(-)}(z_i(x))$	0.5	0.5	0.5	0.7	0.7	0.7	0.6	0.6	0.6
Solutions by Angelov's technique		*No feasible solution found*			*No feasible solution found*			*No feasible solution found*		
I-Pareto optimal solutions		15.833	18.333	27.333	15.829	22.829	24.129	15.734	21.034	25.034
Pareto optimal solutions		15.833	18.333	27.333	15.829	22.829	24.129	15.734	21.034	25.034
Optimal grades of	$T_i^{(+)}(z_i(x))$	1.56	1.56	1.56	0.34	0.34	0.34	-0.58	-0.58	-0.58
	$T_i^{(-)}(z_i(x))$	-0.33	-0.33	-0.33	0.47	0.47	0.47	1.06	1.06	1.06
Remarks		1. Removal of $\alpha+\beta\le 1, \beta\ge 0$ (and not adding $\alpha\le 1$) gives optimal solution. 2. *More acceptable optimal values* than crisp method to each objective function [see, Gao et al 2003]			1. Removal of $\alpha\ge\beta$ gives optimal solution. 2. *More acceptable optimal values* than crisp method to each objective function [see, Gao et al 2003]			1. Removal of $\alpha\le 0$ (and not adding $\beta\ge 1$) gives optimal solution. 2. *More acceptable optimal values* than crisp method to each objective function [see, Gao et al 2003]		

subject to

$$\begin{aligned}
&z_1(x)\le 16.373,\\
&1.2x_{11}+0.9x_{31}+x_{41}+1.1x_{51}+0.95x_{71}\ge 60,\\
&1.25x_{12}+0.95x_{22}+1.15x_{32}+1.05x_{62}\ge 30,\\
&1.3x_{33}+1.1x_{53}\ge 10,\ 1.12x_{14}+1.24x_{34}\ge 70,\\
&2x_{12}+2x_{22}+2x_{32}+2x_{62}+3x_{33}+3x_{53}=x_{11}+x_{31}+x_{41}+x_{51}+x_{71}+x_{14}+x_{34},\\
&x_{ij}\ge 0, \forall i=1\ldots 7,\ j=1\ldots 4.
\end{aligned}$$

First individual maximum and minimum of each objective function $z_i(x), i=1,2,3$ are computed. Based on those values, DM suggests goals and tolerances of each objective function $z_i(x), i=1,2,3$ as given in Table 8. Again using these goals, tolerances; strictly monotonic decreasing $T^{(+)}$-characteristic function and strictly monotonic increasing $T^{(-)}$-characteristic function of each objective function $z_i(x), i=1,2,3$ are constructed. Different cases may be considered based on different goals and tolerances (assuming that these are supplied by DM or presumed values) and the results are tabulated in table 8.

On Table 8, in case I, case II as well as case III, it is shown that there exists no optimal solution if usual membership and non-membership functions are used and if existing techniques are followed in optimization technique under IF environment. But if strictly monotonic $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions are used, proposed algorithm generates I-Pareto optimal solutions and hence Pareto optimal solutions. Moreover the Pareto optimal solutions in case I, case II and case III in Table 8 are more acceptable to the DM than crisp solution as given by Gao (2003). In proposed algorithm, the optimal values of $z_1(x), z_2(x), z_3(x)$ in case I, case II and case III are Pareto optimal whereas the optimal values by Gao (2003) are not. And result in case I is better than solution number 2, case II is better than solution number 3and solution number 4, case III is better than solution number 1 in crisp optimization by Gao (2003).

5.3.2. Another set of $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions

It is discussed in this paper that traditional membership function is not the only function whose higher value gives better value of objective function. Similar statement holds for non-membership function as well. Now $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions in example 5.3 are constructed analogous to traditional membership and non-membership functions. But for minimizing type of objective functions $z_i(x)$, one example of $T^{(+)}$-characteristic function may be $-z_i(x)$ and of $T^{(-)}$-characteristic function may be $z_i(x)$ itself. These are useful specially when goals and/or tolerances cannot be defined easily by DM or even a DM may not be present at all; so that traditional membership and/or non-membership functions cannot be defined for IF objectives. Using $-z_i(x)$ as $T^{(+)}$-characteristic function and $z_i(x)$ as $T^{(-)}$-characteristic function for each IF objective $z_i(x)\forall i = 1,2,3$ and applying proposed algorithm on MOLPP (9) under IF environment, single objective optimization problem is obtained as

$$
\begin{aligned}
&\max \quad \alpha - \beta \\
&\textit{subject to} \\
&-z_i(x) \geq \alpha, \forall i = 1,2,3, \\
&z_i(x) \leq \beta, \forall i = 1,2,3, \\
&\mathrm{z}_1(x) \leq 16.373, \\
&1.2x_{11} + 0.9x_{31} + x_{41} + 1.1x_{51} + 0.95x_{71} \geq 60, \\
&1.25x_{12} + 0.95x_{22} + 1.15x_{32} + 1.05x_{62} \geq 30, \\
&1.3x_{33} + 1.1x_{53} \geq 10,\ 1.12x_{14} + 1.24x_{34} \geq 70, \\
&2x_{12} + 2x_{22} + 2x_{32} + 2x_{62} + 3x_{33} + 3x_{53} = x_{11} + x_{31} + x_{41} + x_{51} + x_{71} + x_{14} + x_{34}, \\
&x_{ij} \geq 0, \forall i = 1\ldots 7,\ j = 1\ldots 4, \alpha, \beta \text{ unrestricted in sign.}
\end{aligned}
$$

Using Lingo 15.0.32, I-Pareto optimal solutions are obtained as follows

$$\alpha = -23.704,\ \beta = 23.704,\ z_1(x^*) = 16.373,\ z_2(x^*) = 23.704, z_3(x^*) = 23.704.$$

To test the Pareto optimality of these I-Pareto optimal solutions, following problem is now solved

$$
\begin{aligned}
&\max \sum_{i=1}^{3} \Omega_i + \sum_{i=1}^{3} \mho_i \\
&\textit{subject to} \\
&-z_1(x) - \Omega_1 \geq -16.373,\ z_1(x) + \mho_1 \leq 16.373, \\
&-z_2(x) - \Omega_2 \geq -23.70491,\ z_1(x) + \mho_2 \leq 23.70491, \\
&-z_3(x) - \Omega_3 \geq -23.70491,\ z_1(x) + \mho_3 \leq 23.70491, \\
&\mathrm{z}_1(x) \leq 16.373, 1.2x_{11} + 0.9x_{31} + x_{41} + 1.1x_{51} + 0.95x_{71} \geq 60, \\
&1.25x_{12} + 0.95x_{22} + 1.15x_{32} + 1.05x_{62} \geq 30, \\
&1.3x_{33} + 1.1x_{53} \geq 10,\ 1.12x_{14} + 1.24x_{34} \geq 70, \\
&2x_{12} + 2x_{22} + 2x_{32} + 2x_{62} + 3x_{33} + 3x_{53} = x_{11} + x_{31} + x_{41} + x_{51} + x_{71} + x_{14} + x_{34}, \\
&x_{ij} \geq 0, \forall i = 1\ldots 7,\ j = 1\ldots 4, \Omega_i, \mho_i \geq 0,\ \forall \mathrm{i=1,2,3.}
\end{aligned}
$$

Using Lingo 15.0.32, Pareto optimal solutions $\bar{x}$, $\bar{\Omega}$ and $\bar{\mho}$ are obtained as follows

$$\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}_3 = 0 = \bar{\mho}_1 = \bar{\mho}_2 = \bar{\mho}_3, z_1(\bar{x}) = 16.373,\ z_2(\bar{x}) = 23.704, z_3(\bar{x}) = 23.704.$$

This is another set of Pareto optimal solutions. Moreover instead of these $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions, different type of functions may be constructed as necessary.

6 Conclusion

Technique to find Pareto optimal solutions to multiple objective linear programming problems under IF environment is introduced in this paper. IF technique is one of the richest apparatus for formulation of optimization problems under imprecise environment, thereby generating more satisficing result than crisp optimization technique. It is shown in this paper that all of the constraints in existing techniques are not necessary. Presence of those constraints may make a problem infeasible. These constraints were necessary due to definition of membership and non-membership functions of IF goals. Consequently membership and non-membership functions of IF goals cannot be used in optimization technique under IF environment. Strictly monotonic $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions are proposed to be used in place of existing membership and non-membership functions of IF set.

Moreover, in multiple objective optimizations, Pareto-optimality of solutions guarantees rationality of the decision. One new technique for finding Pareto optimal solutions to MOLPP under IF environment is introduced and discussed in this paper. Hence the proposed algorithm generates I-Pareto optimal solutions, followed by Pareto optimal solutions to MOLPP with IF goals.

As shown in the application, proposed algorithm generates more acceptable solutions than crisp optimization technique by Gao (2003). And this is a general procedure to obtain an I-Pareto-optimal solution that has the additional property of being Pareto-optimal. Proposed algorithm can find Pareto optimal solutions to MOLPP with IF goals even when goals and/or tolerances cannot be set by using $T^{(+)}$-characteristic function and $T^{(-)}$-characteristic function, as shown in example 5.3.2.

From this it may further be concluded that, under these circumstances, the issue of getting no solution even by using IF set theory yielded from several published methods seems worthwhile or even necessary to reconsider. The same is also true for nonlinear problems of this type.

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